Continuity and Differentiability

Multiple Choice Questions

Q: 1 **The function *f* : R -> R is defined by:**

$$f(x) = \begin{cases} |\sin x|, \text{ when } x \text{ is rational} \\ -|\sin x|, \text{ when } x \text{ is irrational} \end{cases}$$

Read the statements carefully and then choose the option that correctly describes them.

***Statement 1* : *f* (*x*) is continuous at *x* = 0.**

***Statement 2* : *f* (*x*) is discontinuous for all *x* ∈ R - {0}.**

1. Statement 1 is true but Statement 2 is false.
2. Statement 1 is false but Statement 2 is true.
3. Both Statement 1 and Statement 2 are true.
4. Both Statement 1 and Statement 2 are false.

Q: 2 **The Signum function *f* : R -> R, defined below, is discontinuous at *x* = 0. The identity function *g* : R -> R, defined below, is continuous everywhere.**

$$f(x) = \begin{cases} 1, \text{ for } x > 0 \\ 0, \text{ for } x = 0 \\ -1, \text{ for } x < 0 \end{cases}$$

$$g(x) = x, \text{ for all } x \in \mathrm{R}$$

Which of these is true about the product of two functions (*f*.*g*)?

1. ($f.g$) is discontinuous at x = 0 but continuous elsewhere in R.
2. ($f.g$) is not differentiable anywhere in R.
3. ($f.g$) is differentiable everywhere in R.
4. ($f.g$) is continuous everywhere in R.

Q: 3 **Read the statements carefully and choose the option that correctly describes them.**

Statement 1: The function $\sqrt{x-5}$ is continuous in $(5, \infty)$.

Statement 1: The function $\sqrt{x-5}$ is differentiablein $(5, \infty)$.

1. Both statements are true and statement 2 explains why statement 1 is true.
2. Both statements are true but statement 2 does NOT explain why statement 1 is true.
3. Statement 1 is true but statement 2 is false.
4. Statement 1 is false but statement 2 is true.

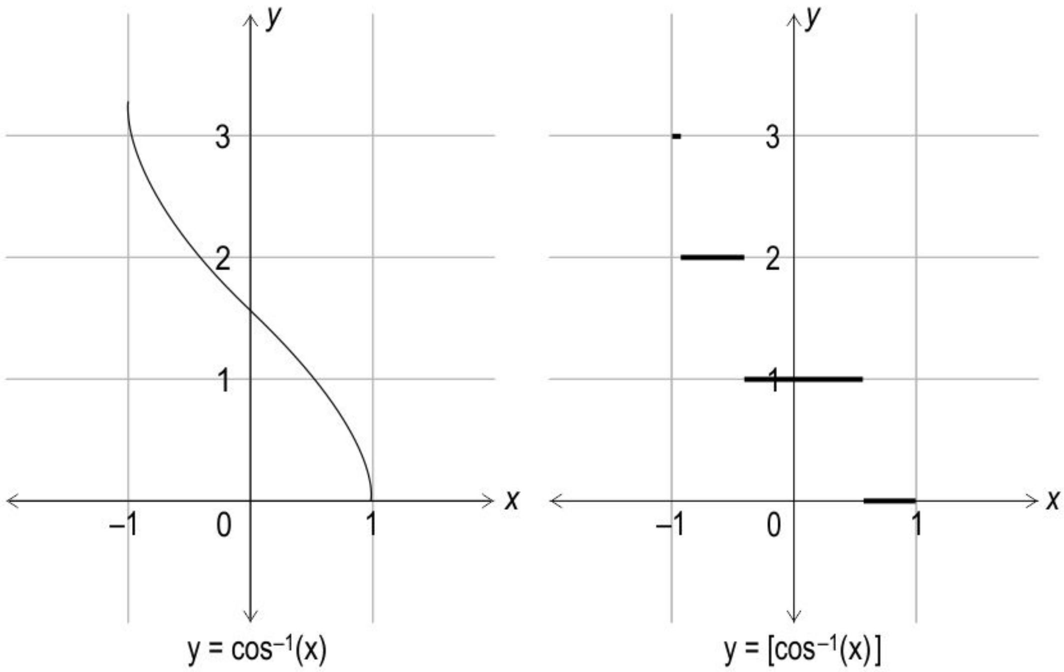
Free Response Questions

Q: 4 **Look at an inverse function below.** **[1]**

$y = \text{cosec}^{-1}(4x^4)$; $|4x^4| > 1$

Find $\frac{dy}{dx}$. Show your steps.

Q: 5 **Shown below are the graphs of two functions $y = \cos^{-1} x$ and $y = [\cos^{-1} x]$, where $[\cos^{-1} x]$ denotes the greatest integer function.** **[2]**



Find the points of discontinuity of the function $y = [\cos^{-1} x]$. Show your work.

Q: 6 If $y = ax^{n+2} + \frac{b}{x^{n+1}}$, where $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$, **[3]**

prove that $x^2y'' = (n+1)(n+2)y$.

Q: 7 **Find the value of $(f \circ g)'$ at $x = 4$ if $f(u) = u^3 + 1$ and $u = g(x) = \sqrt{x}$. Show your work.** **[3]**

Q: 8 **[5]**

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{10}}{10!}$$

If $f(x)$ is differentiated successively 10 times, what is $f^{(10)}(x)$? Show your work.

Q: 9 **[5]**

Find $\frac{d^2y}{dx^2}$ if, $y = tan^{-1}\left[\frac{log\left(\frac{e}{x^5}\right)}{log\left(ex^5\right)}\right] + tan^{-1}\left[\frac{4 + 5\ log\ x}{1 - 20\ log\ x}\right]$.

Show your work.

(*Note* : *Consider the base of logarithm to be* e.)

Q: 10 **Find $\frac{dm}{dn}$ at (m, n) = (-1, 1) where:** **[5]**

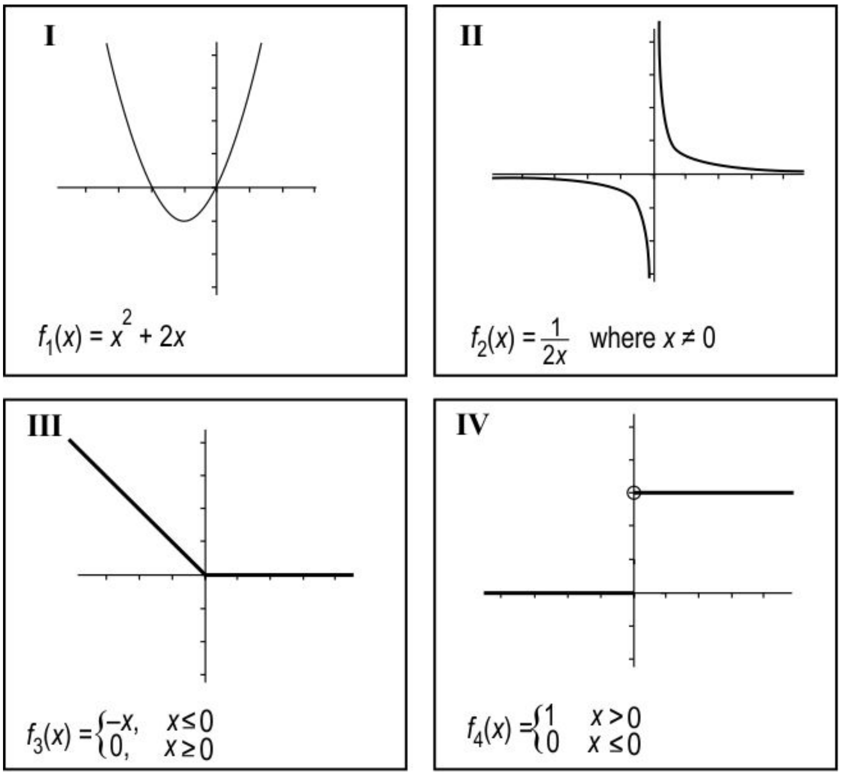
$$5m^2 + 2m - n^{\frac{-2}{3}} = 0$$

Show your work.

Case Study

Answer the questions based on the information given below.

Ambika, a mathematics teacher is conducting a practice session on calculus where she is discussing continuity and differentiability of several functions in their domains. Shown below are four cards that contain a function each along with their domains and graphs.



While discussing in the group, four students claimed as follows:

♦ Leela: "As the function on card I is both continuous and differentiable, we can say that every continuous function is differentiable."
♦ Irfan: "As the graph is not in one piece, the function on card II is discontinuous."
♦ Deepak: "The function on card III is continuous."
♦ Kiran: "The function on card IV is discontinuous."

Q: 11 **Check whether Deepak and Kiran's claims are correct. Justify your answer.** **[2]**

Q: 12 **Is Irfan's claim correct? Does the reason given by him support his claim? Justify.** **[1]**

Q: 13 **Is Leela's claim true for all continuous functions? Justify with a valid reason or provide a counterexample.** **[2]**

The table below gives the correct answer for each multiple-choice question in this test.

Q.No	Correct Answers
1	4
2	4
3	1

Q.No	What to look for	Marks
4	**Differentiates the given function as:** $$\frac{dy}{dx}=\frac{-1}{4x^4\sqrt{(4x^4)^2-1}}\frac{d(4x^4)}{dx}$$	0.5
	Simplifies the above equation as: $$\frac{dy}{dx}=\frac{-16x^3}{4x^4\sqrt{(4x^4)^2-1}}=\frac{-4}{x\sqrt{16x^8-1}}$$	0.5
5	**With the help of the graphs, finds the values of x for which $\cos^{-1} x$ attains integer values as:** $\cos^{-1} x = 3$ $\Rightarrow x = \cos 3$ $\cos^{-1} x = 2$ $\Rightarrow x = \cos 2$ $\cos^{-1} x = 1$ $\Rightarrow x = \cos 1$	1.5
	Concludes that the points of discontinuity of the function $y = [\cos^{-1} x]$ are cos 3, cos 2 and cos 1.	0.5
6	**Finds the first derivative of y as:** $$y' = a(n+2)x^{n+1} - b(n+1)x^{-n-2}$$	1
	Finds the second derivative of y as: $$y'' = a(n+2)(n+1)x^{n} + b(n+1)(n+2)x^{-n-3}$$	1

Q.No	What to look for	Marks
	Multiplies x^2 on both sides of the above equation to get: $x^2y'' = (n+1)(n+2)\left[ax^{n+2} + \frac{b}{x^{n+1}}\right]$ $\Rightarrow x^2y'' = (n+1)(n+2)y$	1
7	**Finds (*f* o *g*)(*x*) as:** $(f \circ g)(x)$ $= f(g(x))$ $= f(\sqrt{x})$ $= (\sqrt{x})^3 + 1$ $= x^{\frac{3}{2}} + 1$	1
	Finds the derivative of (*f* o *g*)(*x*) as: $\frac{d}{dx}(f \circ g)(x)$ $= \frac{d}{dx}(x^{\frac{3}{2}} + 1)$ $= \frac{3}{2}x^{\frac{1}{2}}$	1
	Uses the above step to find the value of (*f* o *g*)' at *x* = 4 as 3.	1

Q.No	What to look for	Marks
8	**Differentiates *f* (*x*) once to get:** $f^{(1)}(x) = 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^9}{9!}$	1
	Rewrites $f^{(1)}$ (*x*) as: $f^{(1)}(x) = f(x) - \frac{x^{10}}{10!}$	0.5
	Finds the second derivative of *f* (*x*) as: $f^{(2)}(x) = f(x) - \frac{x^{10}}{10!} - \frac{x^9}{9!}$	1
	Finds the third derivative of *f* (*x*) as: $f^{(3)}(x) = f(x) - \frac{x^{10}}{10!} - \frac{x^9}{9!} - \frac{x^8}{8!}$	1
	Generalises the pattern as: $f^{(10)}(x) = f(x) - \frac{x^{10}}{10!} - \frac{x^9}{9!} - \cdots - \frac{x^2}{2!} - x$ **(Award full marks if the expression is correctly generalised at the second derivative stage.)**	1

Q.No	What to look for	Marks
	Finds the value of $f^{(10)}(x)$ as $f^{(10)} = 1$.	**0.5**
9	Simplifies $\log\left(\frac{e}{x^5}\right)$ as $1 - 5\log x$.	**0.5**
	Simplifies $\log(ex^5)$ as $1 + 5\log x$.	**0.5**
	Rewrites the given equation as: $y = \tan^{-1}\left[\frac{1-5\log x}{1+5\log x}\right] + \tan^{-1}\left[\frac{4+5\log x}{1-20\log x}\right]$	**0.5**
	Uses the trigonometric identities $\tan^{-1}A - \tan^{-1}B = \tan^{-1}\frac{A-B}{A+B}$ and $\tan^{-1}A + \tan^{-1}B = \tan^{-1}\frac{A+B}{1-AB}$ and rewrites the above equation as: $y = \tan^{-1}1 - \tan^{-1}(5\log x) + \tan^{-1}4 + \tan^{-1}(5\log x)$	**2**
	Simplifies above step to get: $y = \tan^{-1}1 + \tan^{-1}4$	**0.5**
	Differentiates the above equation to get $y' = 0$.	**0.5**
	Differentiates y' to get $y'' = 0$.	**0.5**
10	**Differentiates the equation given in the question with respect to n as follows:** $\frac{d}{dn}\left(5m^2 + 2m - n^{\frac{-2}{3}}\right) = \frac{d}{dn}(0)$ $\Rightarrow \frac{d}{dn}(5m^2) + \frac{d}{dn}(2m) - \frac{d}{dn}\left(n^{\frac{-2}{3}}\right) = 0$	**1**
	Simplifies the differentiation in the previous step as follows: $\frac{d}{dm}(5m^2)\frac{dm}{dn} + 2\frac{dm}{dn} + \frac{2}{3}n^{\frac{-5}{3}} = 0$	**1**

Q.No	What to look for	Marks
	Simplifies the previous step as: $\frac{dm}{dn}(10m + 2) + \frac{2}{3}n^{\frac{-5}{3}} = 0$ **(Award full marks if equivalent solving techniques without some intermediate steps are followed.)**	**1**
	Concludes that: $\frac{dm}{dn} = -\frac{n^{\frac{-5}{3}}}{15m+3}$	**1**
	Finds the value of $\frac{dm}{dn}$ at (-1,1) as $\frac{1}{12}$.	**1**
11	**Writes that Deepak is right and justifies it as follows:** **As f_3 (x) satisfies all the 3 conditions given below, it is continuous.** **♦ f_3 (x) is defined for all real numbers.** **♦ At x = 0, left hand limit = right hand limit = 0** **♦ f (0) = 0**	**1.5**
	Writes that Kiran is right and justifies it as follows: **As f_4 (x) fails to satisfy that at x = 0, left hand limit (0) is not equal to the right hand limit (1), it is discontinuous.**	**0.5**
12	**Writes that the claim made by Irfan is wrong and the reason given by him does not support his claim.**	**0.5**
	Justifies that when x = 0, the function is not defined. So, $\frac{1}{2x}$ is continuous at all points except when x = 0. Hence, writes that the reason given by Irfan is not correct.	**0.5**
13	**Writes that Leela's claim is not true for all continuous functions.**	**0.5**

Q.No	What to look for	Marks
	Gives an example. Foe example: $f_3(x)$ is continuous, but not differentiable as it it has a corner point. **(Award full marks if any other valid examples are given.)**	**1.5**